Vacuum structure and boundary renormalization group

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Abstract
The vacuum structure is probed by boundary conditions. The behaviour of thermodynamical quantities such as free energy, boundary entropy and entanglement entropy under the boundary renormalization group flow are analysed in 2D conformal field theories. The results show that wherever vacuum energy and boundary entropy turn out to be very sensitive to boundary conditions, the vacuum entanglement entropy is independent of boundary properties when the boundary of the entanglement domain does not overlap the boundary of the physical space. In all cases the second law of thermodynamics holds along the boundary renormalization group flow.

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1. Introduction

The vacuum of a quantum field theory has a very rich structure which in fact encodes the complete information about the whole quantum theory. Then, the response of the vacuum to an external perturbation is a very rich source of information not only about its own structure but also about the very nature of the quantum field theory. A particularly interesting perturbation is introduced by the confinement of the quantum system to a finite space volume. In such a case, the effect of boundary conditions into the structure of the vacuum gives rise to interesting phenomena such as spontaneous symmetry breaking and other low energy phenomena.

It is well known that the vacuum energy is very sensitive to the choice of boundary conditions. This dependence generates interesting physical phenomena, such as changes of sign in the Casimir energy. At finite temperature boundary conditions do not only affect the vacuum but also thermodynamic quantities such as the free energy and entropy. In recent years some new quantities have been introduced to measure the entanglement structure of the quantum vacuum [1] (entanglement entropy) and the structure of boundary states [2] (boundary entropy). The evolution of the new types of entropy seems to confirm the validity

of the second law of thermodynamics beyond the natural realm of thermodynamics. One way of testing the monotone behaviour of the new entropies in quantum field theory is to introduce an external perturbation and to analyse their evolution. In this paper we analyse the behaviour of those quantities under the change of boundary conditions. In particular, we consider the effect of such a perturbation on a conformal invariant theory and we analyse the behaviour of the thermodynamical quantities under the induced boundary renormalization group flow.

2. Conformal invariance and boundary renormalization group flow

Let us consider a single real massless scalar field defined in $[0, L]$. From a classical viewpoint there is a large class of boundary conditions which can be imposed on the fields. However, in the quantum theory unitarity and causality impose severe constraints on the boundary behaviour of quantum fields. In particular, the Hamiltonian with density

$$\mathcal{H} = -\frac{1}{2}\frac{\delta^2}{\delta\phi^2} + \frac{1}{2}\phi\sqrt{-\Delta}\phi \tag{1}$$

has to be self-adjoint, which requires that the boundary conditions have to preserve the positivity of the Laplace–Beltrami operator $-\Delta$. The set of boundary conditions which are compatible with unitarity and causality span a four-dimensional manifold which contains the following two families of boundary conditions which parametrize two complete charts of the corresponding four-dimensional manifold [3]

 (a) Mixed boundary conditions

$$L\varphi'(0) = -a\varphi(0) - b\varphi(L); \qquad L\varphi'(L) = b\varphi(0) + d\varphi(L) \tag{2}$$

with $ad - b^2 > 0$ and $a, d \geqslant 0$, which interpolate between Neumann ($a = b = d = 0$) and Dirichlet ($a = b = d = \infty$) boundary conditions; and
 (b) Closed boundary conditions

$$\varphi(L) = \alpha\varphi(0) + \beta L\varphi'(0); \qquad L\varphi'(L) = \gamma\varphi(0) + \delta L\varphi'(0) \tag{3}$$

with $\alpha\delta + \gamma\beta = 1$ and $\alpha\gamma \geqslant 0$, $\beta\delta \geqslant 0$, which include quasi-periodic boundary conditions ($\beta = \gamma = 0$) that interpolate between periodic ($\alpha = \delta = 1$, $\beta = \gamma = 0$) and antiperiodic ($\alpha = \delta = -1$, $\beta = \gamma = 0$) boundary conditions. Note that Zaremba boundary conditions

$$\varphi(0) = 0, \varphi'(L) = 0$$

can be considered either as mixed boundary conditions with $a = \infty, b = d = 0$ or as quasi-periodic boundary conditions with $\alpha = \infty, \gamma = \delta = 0$.

Although the theory is massless, conformal invariance may be broken by the effect of boundary conditions [2]. The only boundary conditions which preserve conformal invariance are Neumann, Dirichlet and quasi-periodic boundary conditions [4]. All other boundary conditions are not invariant under scale transformations and generate a non-trivial renormalization group flow [5]. Because the fields are non-interacting, this flow is simply given by

$$\partial_t a = -a, \qquad\qquad \partial_t b = -b \qquad \partial_t d = -d \tag{4}$$

$$\partial_t \alpha = \partial_t \delta = 0 \qquad\qquad \partial_t \beta = \beta \qquad \partial_t \gamma = -\gamma \tag{5}$$

where the renormalization group parameter t is defined by $L = L_0\,\mathrm{e}^t$.

The fixed points of this flow correspond to conformal invariant theories $a = b = d = 0$ (Neumann), $\beta = \gamma = 0$ (quasi-periodic) and $a = b = d = \infty$ (Dirichlet). Any boundary condition flows towards one of these fixed points.

Mixed boundary conditions flow with the boundary renormalization group from Dirichlet (UV) towards Neumann (IR) conditions. Critical exponents can be identified with the eigenvalues of the renormalization matrix at the fixed points: all critical exponents are either 1 or −1 which in particular shows that there are not cyclic orbits [4].

The most stable fixed point corresponds to Neumann boundary conditions because all its critical exponents are +1. The most unstable is that of Dirichlet boundary conditions since all critical exponents are −1. Quasi-periodic fixed points present relevant and irrelevant perturbations with critical exponents ±1, respectively. Negative critical exponents point out the instabilities. Implications of these results for string theory are well known [5].

3. Vacuum energy and free energy

The infrared properties of quantum field theory are very sensitive to boundary conditions [6]. In particular, the physical properties of the quantum vacuum, free energy and vacuum energy exhibit a very strong dependence on the type of boundary conditions.

In two dimensions there is an infrared problem which makes the analysis more subtle. The consistency of the quantum theory is not guaranteed due to the well-known infrared problems of massless free bosons which prevent the existence of Goldstone phenomena such as spontaneous breakdown of continuous rigid symmetries [7]. The problem arises because the two-point function is not positive which implies that the fundamental property of Osterwalder–Schrader reflection positivity is not satisfied pointing out the inconsistency of the theory. One intimately related property is the odd normalization properties of the vacuum state

$$\Psi(\phi) = \mathcal{N} \, e^{-\frac{1}{2}(\phi, \sqrt{-\Delta}\phi)}. \tag{6}$$

The problem persists even in finite volumes for any choice of boundary conditions and even becomes even more dramatic because then free massless bosons can have zero modes making the vacuum state not normalizable. One way of solving all these problems is to consider a compactification of the scalar field $\Phi = e^{i\phi/R}$ to a circle of unit radius. In that case the correlators of the compactified field Φ satisfy the reflection positivity requirement and the ground state becomes normalizable even in the zero modes sector.

The existence of the zero modes can be partially solved by the choice of boundary conditions. In fact, the Laplace–Beltrami operator Δ presents zero modes only for the mixed conditions with $a = -b = d$, Neumann or closed boundary conditions with $\alpha = \delta = \beta = 1, \gamma = 0$. The only conformally invariant conditions with zero modes are periodic and Neumann boundary conditions, i.e. the boundary conditions of the closed and open strings.

The free energy of the system at finite temperature $1/T$ with the boundary conditions (2), (3) has the following asymptotic expansion for large volume and low temperature $0 < L \ll T$ [8, 9],

$$-\mathrm{Log}\, Z = f_B L T + f_b T + C\frac{T}{L} + \gamma + \mathcal{O}(1/T), \tag{7}$$

where f_B is the bulk free-energy density, f_b is the boundary energy, and C/L is the Casimir energy. The bulk and boundary terms are UV divergent. The bulk free-energy density f_B corresponds to the infinite volume limit of the free-energy density and, thus, in any regularization it does not depend on boundary conditions. On the contrary, the regularized boundary energy f_b and the Casimir energy are highly dependent on boundary conditions. For

instance, f_b is non-vanishing for Dirichlet or Neumann boundary conditions whereas $f_b = 0$ for periodic boundary conditions, and the Casimir energy [10]

$$C = \frac{\pi}{12} - \pi \min_{n \in \mathbb{Z}} \left[\frac{1}{\pi} \arctan \alpha + n + \frac{1}{4} \right]^2 \tag{8}$$

is α dependent for quasi-periodic boundary conditions. The values and signs of this finite size contribution to the energy are very different for periodic ($\alpha = 1, C = -\pi/6$), antiperiodic ($\alpha = -1, C = \pi/12$) and Zaremba ($\alpha = \infty, C = \pi/48$) [11–16].

For mixed boundary conditions, the Casimir energy interpolates between the values $C = -\pi/24$ for Neumann ($a = b = d = 0$) and $C = -\pi/24$ for Dirichlet ($a = d = \infty$) boundary conditions [4]. Now, since $C > -\pi/24$ for generic Robin boundary conditions with $0 < a = d < \infty$ and $b = 0$ [14, 4], the behaviour of Casimir energy is not monotone along the boundary renormalization group trajectories.

4. Boundary entropy

There is another asymptotic limit of the free energy for low temperature and large volume $0 < T \ll L$ of the form

$$-\text{Log} \, Z = f_B L T - \frac{\pi L}{6T} + \lambda + \mathcal{O}(1/L). \tag{9}$$

The second term corresponds to the Casimir energy for periodic boundary conditions.

There is a similar expansion for the entropy

$$S = (1 - T \partial_T) \log Z = \frac{\pi L}{3T} - \lambda + \mathcal{O}(1/L). \tag{10}$$

and the second term $\lambda = -s_b + \lambda_0$ can be split in two pieces: one universal (independent of the boundary condition of the fields) and another one s_b which is known as boundary entropy [2, 17]. This entropy $s_b = \log g$ can be formally associated with the number of boundary states g [2] but in many cases g is not integer and does not correspond to a mere counting of boundary states [17]. It has been conjectured that the quantities g and s_b evolve with the boundary renormalization group flow in a non-increasing way [17]

$$s_{\text{UV}} - s_{\text{IR}} \geqslant 0, \qquad g_{\text{UV}} - g_{\text{IR}} \geqslant 0$$

as it corresponds to any other type of entropy according to the second law of thermodynamics [17–18]. This conjecture is known as g-theorem and has been verified in many cases although not yet proved [19].

Let us analyse what is the behaviour of g and the boundary entropy s_b along the boundary renormalization group flow.

The boundary entropy can easily be computed for quasi-periodic boundary conditions. The result turns out to be [20]

$$s_b = \log |\alpha - 1| - \frac{1}{2} \log \left(\frac{1}{2} + \frac{\alpha^2}{2} \right); \qquad g = \frac{\sqrt{1 + \alpha^2}}{\sqrt{2} |\alpha - 1|}$$

for $\alpha \neq 1$. Although the quasi-periodic boundary conditions describe a curve of fixed points under the boundary renormalization group flow, and the g-theorem does not impose any requirement on the behaviour of their boundary entropy, it turns out to be monotonically increasing for $\alpha < 1$ and monotonically decreasing for $\alpha > 1$, with a vanishing value at the singular point of periodic boundary conditions $\alpha = 1$. In the particular case of Zaremba

boundary conditions the value $g = 1/\sqrt{2}$ agrees with the decreasing values of the flow interpolating from Dirichlet to Neumann boundary conditions

$$g_D = \frac{1}{2R} > g_Z = \frac{1}{\sqrt{2}} > g_N = R,$$

which is in agreement with the conjectured g-theorem for $R < 1/\sqrt{2}$. For free massless scalars, the boundary entropy presents a behaviour under the boundary renormalization group similar to that of the central charge or the standard bulk entropy.

5. Entanglement entropy

Another type of entropy that can be associated with the vacuum state ψ_0 is the entanglement entropy which measures its degree of entanglement. The entanglement entropy is defined as the entropy of the mixed state generated by integrating out the fluctuating modes of the vacuum state Ψ_0 in a bounded domain $(L/2 - l/2, L/2 + l/2)$ of the physical space $(0, L)$ [1], i.e.

$$\rho_l = \int_{L/2-l/2}^{L/2+l/2} \Psi_0^* \Psi_0. \tag{11}$$

The entropy of this state $S_l = -\text{Tr}\, \rho_l \log \rho_l$, is ultraviolet divergent [21–24], but once regularized scales logarithmically with the length l of the interval of integration and the ultraviolet cut-off ϵ introduced to split apart the domain $(L/2 - l/2, L/2 + l/2)$ and its complement $(0, L/2 - l/2 - \epsilon) \cup (0, L/2 + l/2 + \epsilon, L)$

$$S_l = \frac{1}{3} \log \frac{l}{\epsilon} + \gamma(\epsilon). \tag{12}$$

The coefficient $c_1 = 1/3$ of the logarithmic term in (12) is universal and does coincide with one third of the central charge of the corresponding conformal field theory. It is remarkable that coefficient $c_1 = 1/3$ is also absolutely independent of the choice of boundary condition in $(0, L)$. This can be easily understood as a consequence of the fact that the entanglement entropy is rather associated with the behaviour at the interface between $(L/2 - l/2, L/2 + l/2)$ and its complement $(0, L/2 - l/2) \cup (L/2 + l/2, L)$ which does not depend on the choice of boundary conditions at the edge of the physical space. The finite part $\gamma(\epsilon)$ is highly dependent on the ultraviolet regularization method. If the region where the fluctuations are integrated out reaches the boundary itself, e.g. for $(0, l)$, the entropy has the same asymptotic behaviour [25, 26]

$$S_l = \frac{1}{6} \log \frac{l}{\epsilon} + \log g + \frac{1}{2} \gamma(\epsilon), \tag{13}$$

but with a different coefficient $c_1 = 1/6$ for the asymptotic logarithmic term and a different finite term which is also dependent on the boundary condition and related to the boundary entropy. The behaviour of this quantity along the boundary renormalization group flow is then monotonically similar to that of the boundary entropy. The extra $1/2$ factor in the coefficient of the logarithmic term can be understood by the change on the number of boundary points of the entanglement domain. This coefficient is in fact proportional to the number of connected components of that domain [25], which in this case is reduced from two to one. The same topological behaviour is exhibited by the coefficient of a similar logarithmic term that appears in the asymptotic expansion of the entanglement entropy in $2 + 1$ dimensions, and is proportional to the Euler number of the entanglement domain [27].

6. Conclusions

Zamolodchikov introduced with his c-theorem a very interesting characterization of the loss of information associated with the renormalization group flow in 2D quantum field theories [28]. This behaviour of Zamolodchikov c-function is very reminiscent of that of the standard bulk entropy along such RG flow which is the kernel of the second law of thermodynamics.

The Zamolodchikov c-function is associated with the central charge of the conformal anomaly in conformally invariant field theories. Now, for theories defined in finite volumes with boundaries the central charge also governs the behaviour of the finite size corrections to the free energy. However, our results show that the finite size corrections do not behave monotonically along the boundary renormalization group flow. The renormalized trajectory which flows from the Dirichlet fixed point to the Neumann fixed point describes a family of systems whose free energy first increases and then decreases along the same trajectory.

However, the new concepts of entropy, boundary entropy and entanglement entropy behave along the same trajectories as prescribes the second law of thermodynamics. In all cases that we analysed the boundary entropy decreases along the boundary renormalization group flow, although for quasi-periodic boundary conditions the boundary entropy remains constant.

The behaviour of the entanglement entropy is completely different. It is completely independent of the type of boundary conditions if the domain where the quantum fluctuations of the fields are integrated out does not reach the boundary of the space. However, when the domain where the quantum fluctuations have been averaged out reaches the boundary, the entanglement entropy becomes dependent on the boundary conditions, and behaves as prescribes the second law of thermodynamics, decreasing along the boundary renormalization group flow as the boundary entropy does.

Although the analogy between the behaviours of both new types of entropy along the renormalization group flow and that of the standard entropy under time evolution involved the second law of thermodynamics might appear as incidentally due to similar mechanisms of information loss and irreversibility, both behaviours have, in fact, a closer relation from a physical viewpoint. The renormalization group flow of the quantum vacuum or the finite temperature canonical density matrix can also be though as a dynamical adiabatic evolution under the one-parametric family of Hamiltonians connected by scale transformations. Thus, according to the thermodynamical principles, the behaviour of both states under the renormalization group flow must be compatible with the second law. On the other hand this kind of physical time evolution is quite natural from the cosmological point of view, because any physical system is coupled to the underlying background space metric which is continuously expanding towards the infrared along the cosmic time.

The same analysis can also be performed in the presence of bulk interactions, e.g. a $\lambda\phi^4$ term. The renormalization group flow changes by the effect of the perturbation. Some new fixed points appear and some other disappear. In particular, new self-interacting conformal field theories can appear without the infrared problems associated with the free field theories [29]. The analysis of the behaviour of boundary entropy and entanglement entropy in those cases is a challenging open problem.

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